2.- PLANTEAMIENTO DEL MÉTODO DE LOS ELEMENTOS FINITOS.

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1.- INTRODUCCIÓN

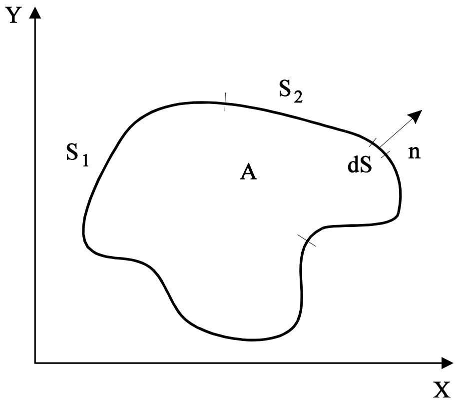
- **METODOLOGÍA PARA LA RESOLUCIÓN NUMÉRICA DE UNA ECUACIÓN DIFERENCIAL EN DERIVADAS PARCIALES MEDIANTE ELEMENTOS FINITOS**

- **APLICACIÓN A LA ECUACIÓN:**

$$D_x \frac{\partial^2 u}{\partial x^2} + D_y \frac{\partial^2 u}{\partial y^2} - G\,u + Q = 0$$

definida en $A \in R^2$, con frontera S

D_x , D_y , G , Q : constantes o función de x e y



- **CONDICIONES DE CONTORNO**

$$\begin{cases} u = \tilde{u} & \text{en } S_1 \\ D_x \dfrac{\partial u}{\partial x} n_x + D_y \dfrac{\partial u}{\partial y} n_y = q & \text{en } S_2 \end{cases}$$

con $\left| \begin{array}{l} S_1 \cap S_2 = \varnothing \quad y \quad S_1 \cup S_2 = S \\ n_x \, , n_y : \text{vector unitario normal al contorno} \end{array} \right.$

2.- RESIDUO

- **SOLUCIÓN ARBITRARIA** $u^*(x, y)$
- **RESIDUO:**

$$R(u^*) = D_x \frac{\partial^2 u^*}{\partial x^2} + D_y \frac{\partial^2 u^*}{\partial y^2} - G u^* + Q$$

- **SOLUCIÓN DE ECUACIÓN $\Rightarrow$ ANULA EL RESIDUO**

$$R(u^*) = 0 \quad \text{si } u^* = u$$

$$R(u^*) \neq 0 \quad \text{si } u^* \neq u$$

3.- MÉTODO DE LOS RESIDUOS PONDERADOS

- **FORMA INTEGRAL**

$$W(u) = \int_A \Psi R(u) dA = 0$$

- **SOLUCIÓN DE ECUACIÓN $\Rightarrow$ ANULA LA FORMA INTEGRAL**
- **ANULACIÓN DE FORMA INTEGRAL PARA TODA Ψ $\Rightarrow$ SOLUCIÓN DE ECUACIÓN**

$$\left. \begin{array}{l} W(u) = \int_A \Psi R(u) dA = 0 \\ \text{para toda } \Psi \end{array} \right\} \rightarrow \quad D_x \frac{\partial^2 u}{\partial x^2} + D_y \frac{\partial^2 u}{\partial y^2} - G u + Q = 0$$

- **APLICACIÓN:**

$$W(u) = \int_A \left(D_x \Psi \frac{\partial^2 u}{\partial x^2} + D_y \Psi \frac{\partial^2 u}{\partial y^2} - G \Psi u + \Psi Q \right) dA$$

4.- INTEGRACIÓN POR PARTES

- **FORMA INTEGRAL:**

- $$W(u) = \int_A \left(D_x \Psi \frac{\partial^2 u}{\partial x^2} + D_y \Psi \frac{\partial^2 u}{\partial y^2} - G \Psi u + \Psi Q \right) dA$$

- **ES CONVENIENTE QUE EL ORDEN DE DERIVACIÓN SOBRE u y Ψ SEAN IGUALES**

- **INTEGRACIÓN POR PARTES:**

$$\iint D_x \Psi \frac{\partial^2 u}{\partial x^2} dx\, dy = \left\{ \begin{array}{ll} u = \Psi & du = \dfrac{\partial \Psi}{\partial x} dx \\ dv = \dfrac{\partial^2 u}{\partial x^2} dx & v = \dfrac{\partial u}{\partial x} \end{array} \right\} =$$

$$= -\iint D_x \frac{\partial \Psi}{\partial x} \frac{\partial u}{\partial x} dx\, dy + \int D_x \Psi \frac{\partial u}{\partial x} dy$$

$$\iint D_y \Psi \frac{\partial^2 u}{\partial y^2} dx\, dy = \left\{ \begin{array}{ll} u = \Psi & du = \dfrac{\partial \Psi}{\partial y} dy \\ dv = \dfrac{\partial^2 u}{\partial y^2} dy & v = \dfrac{\partial u}{\partial y} \end{array} \right\} =$$

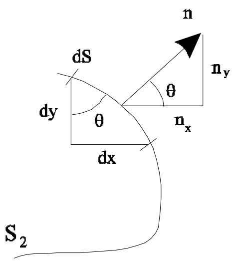
$$= -\iint D_y \frac{\partial \Psi}{\partial y} \frac{\partial u}{\partial y} dx\, dy + \int D_y \Psi \frac{\partial u}{\partial y} dx$$

- **INTEGRALES DE CONTORNO:**

$$\int D_x \Psi \frac{\partial u}{\partial x} dy + \int D_y \Psi \frac{\partial u}{\partial y} dx = \int \Psi \left(D_x \frac{\partial u}{\partial x} dy + D_y \frac{\partial u}{\partial y} dx \right)$$

4.- INTEGRACIÓN POR PARTES

- **INTEGRALES DE CONTORNO:**



$$dx = dS\,\text{sen}\,\theta = n_y dS$$
$$dy = dS\cos\theta = n_x dS$$

$$\int D_x \Psi \frac{\partial u}{\partial x} dy + \int D_y \Psi \frac{\partial u}{\partial y} dx = \int_S \Psi \left(D_x \frac{\partial u}{\partial x} n_x + D_y \frac{\partial u}{\partial y} n_y \right) dS$$

- **ELEGIMOS Ψ PARA QUE SE ANULE EN S_1**

$$\int_S \Psi \left(D_x \frac{\partial u}{\partial x} n_x + D_y \frac{\partial u}{\partial y} n_y \right) dS = \int_{S_2} \Psi \left(D_x \frac{\partial u}{\partial x} n_x + D_y \frac{\partial u}{\partial y} n_y \right) dS$$

- **POR CONDICIÓN DE CONTORNO EN S_2**

$$\int_S \Psi \left(D_x \frac{\partial u}{\partial x} n_x + D_y \frac{\partial u}{\partial y} n_y \right) dS = \int_{S_2} \Psi q \, dS$$

- **FORMA INTEGRAL FINAL:**

$$W(u) = -\int_A \left(D_x \frac{\partial \Psi}{\partial x}\frac{\partial u}{\partial x} + D_y \frac{\partial \Psi}{\partial y}\frac{\partial u}{\partial y} + \Psi G u - \Psi Q \right) dA + \\ + \int_{S_2} \Psi q \, dS$$

5.- DISCRETIZACIÓN

- **DISCRETIZACIÓN:**

$$\bigcup_{e=1}^{Ne} A^{e} = A \quad ; \qquad A^{e_i} \cap A^{e_j} = \varnothing$$

- **FORMA INTEGRAL**

$$W(u) = \sum_{e=1}^{Ne} W^{e}(u)$$

$$W^{e}(u) = -\int_{A^e} \left(D_x \frac{\partial \Psi}{\partial x}\frac{\partial u}{\partial x} + D_y \frac{\partial \Psi}{\partial y}\frac{\partial u}{\partial y} + \Psi G u - \Psi Q \right) dA + \\ + \int_{S_2 \cap S^e} \Psi q \, dS$$

SERÁ NECESARIO CONSIDERAR CONDICIONES DE CONTINUIDAD ENTRE ELEMENTOS.

- **FUNCIONES DEFINIDAS LOCALMENTE EN ELEMENTO**

$$W^{e}(u) = -\int_{A^e} \left(D_x \frac{\partial \Psi^e}{\partial x}\frac{\partial u^e}{\partial x} + D_y \frac{\partial \Psi^e}{\partial y}\frac{\partial u^e}{\partial y} + \Psi^e G u^e - \Psi^e Q \right) dA + \\ + \int_{S_2 \cap S^e} \Psi^e q \, dS$$

6.- APROXIMACIÓN NODAL

- **INTERPOLACIÓN DE ELEMENTOS FINITOS:**

$$u^e(x,y) = \sum_{i=1}^{n} N_i(x,y)\,u^e(x_i,y_i) \qquad \leftrightarrow \qquad u^e(x,y) = [N]\,\{u^e\}$$

CONDICIONES DE CONTINUIDAD ENTRE ELEMENTOS.

7.- FUNCIONES DE PONDERACIÓN. FORMULACIÓN DE GALERKIN

- **GALERKIN:**

$$\Psi^e(x,y) = \sum_{i=1}^{n} N_i(x,y)\,\Psi^e(x_i,y_i) \qquad \leftrightarrow \qquad \Psi^e(x,y) = [N]\,\{\Psi^e\}$$

8.- PLANTEAMIENTO MATRICIAL DE LA FORMA INTEGRAL

- **FORMA INTEGRAL**

$$W(u) = \sum_{e=1}^{Ne} W^e(u)$$

$$W^e(u) = -\int_{A^e} \left(D_x \frac{\partial \Psi^e}{\partial x}\frac{\partial u^e}{\partial x} + D_y \frac{\partial \Psi^e}{\partial y}\frac{\partial u^e}{\partial y} + \Psi^e G u^e - \Psi^e Q \right) dA +$$
$$+ \int_{S_2 \cap S^e} \Psi^e q \, dS$$

$$W^e(u) = W_1^e(u) + W_2^e(u) + W_3^e(u) + W_4^e(u)$$

$$W_1^e(u) = -\int_{A^e} \left(D_x \frac{\partial \Psi^e}{\partial x}\frac{\partial u^e}{\partial x} + D_y \frac{\partial \Psi^e}{\partial y}\frac{\partial u^e}{\partial y} \right) dA$$

$$W_2^e(u) = -\int_{A^e} G \Psi^e u^e \, dA$$

$$W_3^e(u) = \int_{A^e} Q \Psi^e \, dA$$

$$W_4^e(u) = \int_{S_2 \cap S^e} \Psi^e q \, dS$$

- **DEFINICIONES:**

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \quad ; \quad [D] = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix} \quad ; \quad [B] = [L][N]$$

8.- PLANTEAMIENTO MATRICIAL DE LA FORMA INTEGRAL

- **DESARROLLO: W_1**

$$W_1^e(u) = -\int_{A^e} \left(D_x \frac{\partial \Psi^e}{\partial x}\frac{\partial u^e}{\partial x} + D_y \frac{\partial \Psi^e}{\partial y}\frac{\partial u^e}{\partial y} \right) dA =$$

$$= -\int_{A^e} \left(\left([L]\Psi^e \right)^T \, [D] \, \left([L]\, u^e \right) \right) dA$$

- **APROXIMACIÓN NODAL:**

$$u^e(x,y) = [N]\,\{u^e\} \qquad \Psi^e(x,y) = [N]\,\{\Psi^e\}$$

$$W_1^e(u) = -\int_{A^e} \left(\left([L][N]\{\Psi^e\} \right)^T \, [D] \, \left([L][N]\{u^e\} \right) \right) dA$$

$$W_1^e(u) = -\{\Psi^e\}^T \left\{ \int_{A^e} \left([B]^T \, [D] \, [B] \right) dA \right\} \{u^e\} =$$

$$= \{\Psi^e\}^T \, [k_1^e] \, \{u^e\}$$

$$[k_1^e] = -\int_{A^e} \left([B]^T \, [D] \, [B] \right) dA$$

8.- PLANTEAMIENTO MATRICIAL DE LA FORMA INTEGRAL

- **DESARROLLO: W$_2$**

$$W_2^e(u) = -\int_{A^e} G\,\Psi^e\,u^e\,dA$$

  - **APROXIMACIÓN NODAL**

$$u^e(x,y) = [N]\,\{u^e\} \qquad \Psi^e(x,y) = \{\Psi^e\}^T[N]^T$$

$$W_2^e(u) = -\{\Psi^e\}^T \left\{ \int_{A^e} G[N]^T[N]\,dA \right\} \{u^e\} = \{\Psi^e\}^T\,[k_2^e]\,\{u^e\}$$

$$[k_2^e] = -\int_{A^e} G[N]^T[N]\,dA$$

- **DESARROLLO: W$_3$**

$$W_3^e(u) = \int_{A^e} Q\,\Psi^e\,dA$$

  - **APROXIMACIÓN NODAL**

$$W_3^e(u) = \{\Psi^e\}^T \left\{ \int_{A^e} Q\,[N]^T\,dA \right\} = \{\Psi^e\}^T\,\{f_3^e\}$$

$$\{f_3^e\} = \int_{A^e} Q\,[N]^T\,dA$$

- **DESARROLLO: W$_4$**

$$W_4^e(u) = \int_{S_2 \cap S^e} \Psi^e\,q\,dS$$

  - **APROXIMACIÓN NODAL**

$$W_4^e(u) = \int_{S_2 \cap S^e} \Psi^e\,q\,dS = \{\Psi^e\}^T \left\{ \int_{S_2 \cap S^e} q[N]^T dS \right\} = \{\Psi^e\}^T\{f_4^e\}$$

$$\{f_4^e\} = \int_{S_2 \cap S^e} q[N]^T dS$$

8.- PLANTEAMIENTO MATRICIAL DE LA FORMA INTEGRAL

- **FORMA INTEGRAL DE ELEMENTO:**

$$W^e(u) = W_1^e(u) + W_2^e(u) + W_3^e(u) + W_4^e(u)$$

$$W_1^e(u) = \left\{\Psi^e\right\}^T \left[k_1^e\right] \left\{u^e\right\} \qquad \left[k_1^e\right] = -\int_{A^e} \left([B]^T \; [D] \; [B] \right) dA$$

$$W_2^e(u) = \left\{\Psi^e\right\}^T \left[k_2^e\right] \left\{u^e\right\} \qquad \left[k_2^e\right] = -\int_{A^e} G[N]^T [N] dA$$

$$W_3^e(u) = \left\{\Psi^e\right\}^T \left\{f_3^e\right\} \qquad \left\{f_3^e\right\} = \int_{A^e} Q \; [N]^T dA$$

$$W_4^e(u) = \left\{\Psi^e\right\}^T \left\{f_4^e\right\} \qquad \left\{f_4^e\right\} = \int_{S_2 \cap S^e} q [N]^T dS$$

$$W^e(u) = \left\{\Psi^e\right\}^T \left(\left(\left[k_1^e\right] + \left[k_2^e\right] \right) \left\{u^e\right\} + \left\{f_3^e\right\} + \left\{f_4^e\right\} \right)$$

$$W^e(u) = \left\{\Psi^e\right\}^T \left(\left[k^e\right] \left\{u^e\right\} + \left\{f^e\right\} \right)$$

9.- ENSAMBLADO

- **FORMA INTEGRAL TOTAL:**

$$W(u) = \sum_{e=1}^{Ne} W^e(u) = 0$$

NÚMERO DE NODOS TOTAL = M

$$W^e(u) = \left\{\Psi^e\right\}^T \left(\left[k^e\right] \left\{u^e\right\} + \left\{f^e\right\} \right)$$

SOLO AFECTA AL VALOR EN NODOS DEL ELEMENTO

- **EXPANSIÓN DE MATRICES:**

$$W^e(u) = \{\Psi\}^T \left(\left[K^e\right] \{U\} + \left\{F^e\right\} \right)$$

$$W^e(u) = \sum_{e=1}^{Ne} W^e(u) = 0$$

$$= \sum_{e=1}^{Ne} \left(\{\Psi\}^T \left(\left[K^e\right] \{U\} + \left\{F^e\right\} \right) \right) = 0$$

$$= \{\Psi\}^T \left(\left(\sum_{e=1}^{Ne} \left[K^e\right] \right) \{U\} + \sum_{e=1}^{Ne} \left\{F^e\right\} \right) = 0$$

$$= \{\Psi\}^T \left([K]\{U\} + \{F\} \right) = 0$$

- **ES NECESARIO CONSIDERAR LA ANULACIÓN DE Ψ EN CONTORNO S_1**

10.- CONDICIONES DE CONTORNO ESENCIALES

- **ECUACIÓN FINAL:**

$$\{\Psi\}^T \left([K]\{U\} + \{F\} \right) = 0$$

- **CONDICIONES DE CONTORNO:**
  - **S_2 : YA CONSIDERADAS EN FORMULACIÓN**
  - **S_1 : VALOR DE u ESPECIFICADO EN NODOS:** $u(x_j, y_j) = \tilde{u}_j$ **Y ELIMINACIÓN DE LAS ECUACIONES CORRESPONDIENTES A $\Psi = 0$**

$$\begin{bmatrix} K_{11} & \cdots & K_{1j} & \cdots & K_{1M} \\ \vdots & \ddots & \vdots & & \vdots \\ K_{j1} & \cdots & K_{jj} & \cdots & K_{jM} \\ \vdots & & \vdots & \ddots & \vdots \\ K_{M1} & \cdots & K_{Mj} & \cdots & K_{MM} \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_M \end{Bmatrix} + \begin{Bmatrix} F_1 \\ \vdots \\ F_j \\ \vdots \\ F_M \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} K_{11} & \cdots & K_{1j} & \cdots & K_{1M} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ K_{M1} & \cdots & K_{Mj} & \cdots & K_{MM} \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_M \end{Bmatrix} + \begin{Bmatrix} F_1 \\ \vdots \\ -\tilde{u}_j \\ \vdots \\ F_M \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} K_{11} & \cdots & 0 & \cdots & K_{1M} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ K_{M1} & \cdots & 0 & \cdots & K_{MM} \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_M \end{Bmatrix} + \begin{Bmatrix} F_1 + K_{1j}\tilde{u}_j \\ \vdots \\ -\tilde{u}_j \\ \vdots \\ F_M + K_{Mj}\tilde{u}_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$